On the Margin: The effects of introducing or swapping votes on election margins

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1 Introduction

The margin of victory is the crucial quantity in auditing elections and, in general, in protecting the election outcomes against fraud and/or error.

In the most simple case of a vote-for-one contest, the margin, M, is the difference in votes between the winning candidate, V_w, and the candidate that just lost, V_l, i.e., the candidate that lost but gained the most votes over all other losing candidates:

$$M = V_w - V_l. \tag{1}$$

In contests where multiple candidates can be elected, this quantity is modified such that the winning candidate that won with the least votes is the "winning candidate" for purposes of calculating the smallest margin between winning and losing candidates.

In supermajority elections, where a ballot choice or candidate must win by at least a certain fraction of votes, typically $\frac{1}{2}$ or $\frac{2}{3}$, the margin of interest is a bit different. In this case, we want the margin to be the shortest distance, in votes, between a candidate or choice and the threshold needed to win. This distance is essentially the absolute value of the difference between the highest vote-winning candidate (or the affirmative choice in a ballot measure) and the threshold amount of votes that the office or ballot measure requires to be elected. For example, the margin in a contest that requires winning $\frac{2}{3}$ of votes to win is

$$M = \left| V_w - \frac{2}{3}(V_w + V_l + \ldots) \right|, \tag{2}$$

where the last term is the sum of all valid ballots.

2 Swapping Votes

For simplicity sake, let's examine two two-candidate elections, one with a plurality requirement (the most votes win) and another with a $\frac{2}{3}$ supermajority requirement. The margin in the first race is

$$M_1 = V_w - V_l,$$

and the second is

$$M_2 = \left| V_w - \frac{2}{3}(V_w + V_l) \right|.$$

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2.1 Swapping votes in plurality

In the plurality case, changing a vote from a vote for the winner to a vote for the loser

$$V_w^{'} = V_w - 1, \tag{3}$$

$$V_l^{'} = V_l + 1, \tag{4}$$

reduces the margin by 2 votes

$$M_1^{'} = V_w^{'} - V_l^{'} = (V_w - 1) - (V_l + 1) = \boxed{M_1 - 2}. \tag{5}$$

Changing a vote from a vote for the looser to a vote for the winner

$$V_w^{'} = V_w + 1, \tag{6}$$

$$V_l^{'} = V_l - 1, \tag{7}$$

increases the margin by 2 votes

$$M_1^{'} = V_w^{'} - V_l^{'} = (V_w + 1) - (V_l - 1) = \boxed{M_1 + 2}. \tag{8}$$

2.2 Swapping votes in supermajority

In the supermajority case, changing a vote from a vote for the winner to a vote for the loser, per (3) and (4), changes the margin by 1 vote:

$$\begin{aligned} M_2^{'} &= \left| V_w^{'} - \frac{2}{3}(V_w^{'} + V_l^{'}) \right|, \\ &= \left| (V_w - 1) - \frac{2}{3}((V_w - 1) + (V_l + 1)) \right|, \\ &= \left| V_w - \frac{2}{3}(V_w + V_l) - 1 \right|, \\ &= \boxed{\begin{cases} M_2 - 1, & V_w > \frac{2}{3}V_T \\ M_2 + 1, & V_w \leq \frac{2}{3}V_T \end{cases}}, \end{aligned} \tag{9}$$

where $V_T \equiv V_w + V_l$ is the total number of valid votes (votes cast for valid choices) in the two-candidate example.

And, similarly, changing a vote from the loser to winner, per (6) and (7), also changes the margin by one vote:

$$\begin{aligned} M_2^{'} &= \left| V_w^{'} - \frac{2}{3}(V_w^{'} + V_l^{'}) \right|, \\ &= \left| (V_w + 1) - \frac{2}{3}((V_w + 1) + (V_l - 1)) \right|, \\ &= \left| V_w - \frac{2}{3}(V_w + V_l) + 1 \right|, \\ &= \boxed{\begin{cases} M_2 + 1, & V_w \geq \frac{2}{3}V_T \\ M_2 - 1, & V_w < \frac{2}{3}V_T \end{cases}}. \end{aligned} \tag{10}$$

3 Adding Ballots

In some cases, ballots that a machine reads as showing too many choices in a contest (“overvotes”) or too few choices in a contest (“undervotes”) are in fact valid votes for one candidate or another.

3.1 Adding votes in plurality

If an invalid ballot is found to hold a valid vote for the winner, the totals become

$$V_w^{'} = V_w + 1, \tag{11}$$

$$V_l^{'} = V_l. \tag{12}$$

Which adds one vote to the margin

$$M_1^{'} = V_w^{'} - V_l^{'} = V_w + 1 - V_l = \boxed{M_1 + 1}. \tag{13}$$

Similarly, an invalid ballot that has been found to contain a valid vote for the loser changes the totals as such

$$V_w^{'} = V_w, \tag{14}$$

$$V_l^{'} = V_l + 1. \tag{15}$$

Which subtracts one vote from the margin

$$M_1^{'} = V_w^{'} - V_l^{'} = V_w - V_l - 1 = \boxed{M_1 - 1}. \tag{16}$$

3.2 Adding votes in supermajority

This is where it gets a bit strange. Converting invalid ballots to valid votes in supermajority races, actually affects the margin with *fractional votes.* This is because the new ballot affects both the winning or losing total in the margin calculation *and*, naturally, the combined total.

For example, adding a vote to the winner's totals, per (11) and (12), can change the margin by $\frac{1}{3}$ of a vote:

$$\begin{aligned} M_2^{'} &= \left| V_w^{'} - \frac{2}{3}(V_w^{'} + V_l^{'}) \right|, \\ &= \left| (V_w + 1) - \frac{2}{3}((V_w + 1) + V_l) \right|, \\ &= \left| V_w - \frac{2}{3}(V_w + V_l) - \frac{1}{3} \right|, \\ &= \boxed{\begin{cases} M_2 + \frac{1}{3}, & V_w \geq \frac{2}{3}V_T \\ M_2 - \frac{1}{3}, & V_w < \frac{2}{3}V_T \end{cases}}. \end{aligned} \tag{17}$$

And, adding a vote to the loser's share of votes, per (14) and (15), will change the margin by $\frac{2}{3}$ of a vote:

$$\begin{aligned} M_2^{'} &= \left| V_w^{'} - \frac{2}{3}(V_w^{'} + V_l^{'}) \right|, \\ &= \left| V_w - \frac{2}{3}(V_w + (V_l + 1)) \right|, \\ &= \left| V_w - \frac{2}{3}(V_w + V_l) - \frac{2}{3} \right|, \\ &= \boxed{\begin{cases} M_2 - \frac{2}{3}, & V_w > \frac{2}{3}V_T \\ M_2 + \frac{2}{3}, & V_w \leq \frac{2}{3}V_T \end{cases}}. \end{aligned} \tag{18}$$